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Kinetic Theory Applied to Inclined Flows

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Abstract We apply the continuum equations of a kinetic theory to predict the features of uniform, steady, inclined flows of identical, frictional, inelastic spheres over a rigid, bumpy base between vertical, frictional side walls. Numerical solutions of these equations over a range of mass flow rates exhibit features seen in physical experiments and numerical solutions in the absence of side walls. For the densest flows, we employ a phenomenological extension of kinetic theory that involves a length scale associated with particle correlations. When a dense flow is thick enough, an algebraic balance between the production and dissipation of fluctuation energy reproduces the relation between mass flow rate and mass hold-up obtained when solving the boundary-value problem of the extended theory.

Keywords Kinetic theory · Inclined flow · Inelastic spheres

1 Introduction

Over the past twenty-five years, numerous kinetic theories have been derived for flows of identical, hard, spherical particles that interact through inelastic collisions at moderate volume fractions - typically less than 0.49, at which a transformation to an ordered phase is first possible [1]. These theories are of two types: those linear in the first spatial gradients [2–4] and those of higher order [5–7]. Theories for frictional spheres involve additional balance equations associated with the rotational degrees of freedom [8,9]. Indications are that kinetic theories for flows at higher volume fractions must involve more complicated distributions of relative velocity than are usually employed [10,11]; while phenomenological corrections to existing kinetic theories have been introduced to account for frictional yield [12,**?**], correlated interactions [13,14] or both [15]. Kinetic theories linear in the first spatial gradients have had some success in describing moderately dense, inclined flows observed in physical experiments [12,16–18]. Similarly, the phenomenological extensions provide good descriptions of dense flows, provided that the coefficients they employ are appropriately adjusted [19].

Previously [19], we illustrated the predictive capability of the kinetic theory developed by Garzo and Dufty [3] for identical, frictionless, inelastic spheres in steady, uniform, inclined flows between vertical, flat, frictional side walls over erodible particle beds. Here, we consider such flows over rigid, bumpy beds. As before, for volume fractions roughly greater than 0.49, we extended the theory of Garzo and Dufty by introducing an additional length scale into the expression for the rate of collisional dissipation. This is identified with the length of chains of particles that are experiencing multiple or correlated collisions. It is determined by a balance between the orienting influence of the flow and the randomizing influence of the collisions.

Numerical solutions for the dense flows that incorporate this extension, and a simpler algebraic theory that is based on it, reproduce the relation between mass flow rate and mass hold-up measured in physical experiments. The numerical solutions show that, as the mass hold-up is increased, there are multiple solutions for the same mass flow rate. This is also seen in phys-

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ical experiments over a flat, frictional base [12] and in numerical solutions for inclined flows in the absence of side walls [20].

2 Kinetic Theory

2.1 Balance equations

We consider identical spheres with a mass m and a diameter d. The mean number of particles per unit volume is n, the mass density, ρ, is the product of m and n, the mean velocity $\mathbf{u}$ is the average over the particle velocity $\mathbf{c}$ using the single particle velocity distribution function, $\mathbf{u} \equiv \langle \mathbf{c} \rangle$; the fluctuation velocity is, then, $\mathbf{C} \equiv \mathbf{c} - \mathbf{u}$, and a measure of the strength of the velocity fluctuations, the granular temperature, is defined by $T \equiv \langle \mathbf{C} \cdot \mathbf{C} \rangle /3$.

The mass density, the mean velocity, and the granular temperature are determined as solutions to the balance of mass and linear momentum that have their usual forms [2], and the balance of fluctuation energy,

$$(3/2)\rho\dot{T} = tr\left(\mathbf{tD}\right) - \nabla \cdot \mathbf{q} - \Gamma. \quad (1)$$

In this, the convected time rate of change of internal energy following the mean motion is equal to the rate at which fluctuations are created by the working of the stress $\mathbf{t}$ through the symmetric part $\mathbf{D}$ of the gradients of the mean velocity, less the sum of the divergence of the flux of fluctuation energy $\mathbf{q}$ and the rate of collisional dissipation Γ.

Numerical simulations of homogeneous, steady shearing flows of granular materials [21] indicate that the presence of friction does have an influence on the tractions necessary to maintain a shearing flow at a given solids fraction. Here, we take the frictional interactions into account only through their influence on the energy of the fluctuations in the translational velocity [9].

2.2 Constitutive relations

To incorporate the additional dissipation due to friction, we introduce an effective coefficient of restitution, ε, in the limit of large friction. In this case, Jenkins and Berzi [14] indicate that ε is given in terms of the normal and tangential coefficients of restitution e and β in sticking collisions [24] by

$$\varepsilon^2 \equiv e^2 - 4\frac{1+\beta}{7} + 4\left(\frac{1+\beta}{7}\right)^2\left[1 + \frac{5(1+\beta)}{9-5\beta}\right]. \quad (2)$$

Here, we use an effective coefficient of restitution of 0.60, which corresponds to values of the normal and tangential coefficients of 0.92 and 0.25, which are close to values measured by Foerster, et al. [24] for glass spheres.

For solid volume fractions $\nu \equiv n\pi d^3/6$ less than, roughly, 0.49, we employ the constitutive relations for shearing flows that result from the kinetic theory of Garzo and Dufty [3] for identical, frictionless, inelastic spheres, but do not incorporate the small terms introduced by their function c^* of the coefficient of restitution. The magnitude of c^* is about 0.04 and terms proportional to c^* are typically multiplied by a small numerical coefficient. If the x coordinate is taken in the flow direction, the y coordinate in the direction of shear, and the z coordinate orthogonal to these, the pressure $p \equiv -\left(t_{xx} + t_{yy} + t_{zz}\right)/3$, the shear stress $S \equiv t_{xy}$, the energy flux $Q \equiv q_y$, and the rate of collisional dissipation Γ are given by

$$p = 4\rho GFT, \quad (3)$$

where $G \equiv \nu(1-\nu/2)/(1-\nu)^3$ is the product of ν and the expression for the volume fraction dependence of the radial distribution function at contact determined by Carnahan and Starling [23] in numerical simulations at moderate volume fractions and $F = (1+\varepsilon)/2 + 1/(4G)$;

$$S = \mu u', \quad (4)$$

where the prime denotes a derivative with respect to y and

$$\mu = 2Jpd/\left(5\pi^{1/2}FT^{1/2}\right), \quad (5)$$

with

$$J = \frac{1+\varepsilon}{2} + \frac{\pi}{32G^2}\frac{\left[5+2(1+\varepsilon)(3\varepsilon-1)G\right]}{24-6(1-\varepsilon)^2-5(1-\varepsilon^2)} \times\left[5+4(1+\varepsilon)G\right]; \quad (6)$$

$$Q = -\kappa T' - \eta\nu', \quad (7)$$

where

$$\kappa = Mpd/\left(\pi^{1/2}FT^{1/2}\right), \quad (8)$$

with

$$M = \frac{1+\varepsilon}{2} + \frac{9\pi}{144}\frac{5+3G(1+\varepsilon)^2(2\varepsilon-1)}{(1+\varepsilon)\left[16-7(1-\varepsilon)\right]G^2} \times\left[5+6G(1+\varepsilon)\right], \quad (9)$$

and

$$\eta = \frac{25\pi^{1/2}}{128}\frac{pT^{1/2}}{4FG}\frac{d}{\nu^2}N, \quad (10)$$

with

$$N = \frac{96}{25}\frac{1-\varepsilon}{1+\varepsilon}\frac{\nu}{G}\frac{5+6G(1+\varepsilon)}{16+3(1-\varepsilon)} \times\left\{\frac{20\nu\left(\ln G\right)_\nu\left[5+3G(1+\varepsilon)^2(2\varepsilon-1)\right]}{48-21(1-\varepsilon)} - G\left[1+\nu\left(\ln G\right)_\nu\right](1+\varepsilon)\varepsilon\right\}, \quad (11)$$

in which the subscript indicates a derivative with respect to ν; and

$$\Gamma = \frac{12}{\pi^{1/2}}\frac{\rho G}{d}(1-\varepsilon^2)T^{3/2}. \quad (12)$$

For $\nu > 0.49$, we adopt the radial distribution function for dense aggregates of colliding spheres proposed by Torquato [26], $G = 0.63\nu/(0.60-\nu)$, where the singular value of ν for the dissipative interaction is taken to be 0.60 [10] and the rate of collisional dissipation (12) is modified by replacing d with a length L that is determined in a balance between the orienting influence of the flow and the randomizing influence of collisions [13]:

$$\frac{L}{d} = \frac{1}{2}cG^{1/3}\frac{du'}{T^{1/2}}, \quad (13)$$

where c is a parameter of order one.
The algebraic part of the balance of energy, $Su' - \Gamma = 0$ may be solved for $du'/T^{1/2}$. When this is employed in (13), it gives

$$\frac{L}{d} = \frac{1}{2}\left[\frac{15}{J}\left(1-\varepsilon^2\right)c^2\right]^{1/3} G^{2/9}, \quad (14)$$

where, for dense flows, terms proportional to G^{-1} and G^{-2} in (6) may be ignored and $J = (1+\varepsilon)/2 + (\pi/4)(3\varepsilon - 1)\left(1+\varepsilon^2\right)/\left[24-(1-\varepsilon)(11-\varepsilon)\right]$. When the constitutive relations (4) and (5) are used to express $du'/T^{1/2}$ in terms of the stress ratio S/p and this is employed with the dense limits of J and $F = (1+\varepsilon)/2$ in the algebraic energy balance,

$$\frac{L}{d} = \frac{24}{5\pi}J\frac{1-\varepsilon}{1+\varepsilon}\left(\frac{p}{S}\right)^2 \quad (15)$$

This, used with (14) gives

$$G = \left[\frac{192}{25\pi^{3/2}}\frac{J^2}{c}\frac{1-\varepsilon}{(1+\varepsilon)^2}\left(\frac{p}{S}\right)^3\right]^3. \quad (16)$$

Equation (14) determines L/d in terms of ν; (16) determines ν in terms of S/p or, equivalently, in terms of the tangent of the angle of inclination.

3 Boundary-Value Problem

We next use the balance laws and constitutive relations outlined above to phrase and solve a one-dimensional boundary-value problem for the steady, uniform flow over a rigid, bumpy base inclined at an angle ϕ to the horizontal between flat, vertical side walls. We consider flows in two different chutes: one in which the walls are separated by a distance W of 30 particle diameters, the other in which the walls are separated by a distance of 1400 particle diameters. In both cases, we take the coefficients of restitution of the wall, e_w, to be 0.60 and the coefficient of sliding friction, μ_w, to be 0.15. We assume that the flow is uniform across the chute and treat the frictional resistance and collisional dissipation of the side walls in an average way by introducing a constant force through the thickness of magnitude $2\mu_w p/W$ opposing the flow and a constant rate of dissipation of $(2/\pi)^{1/2}(1-e_w^2)pT^{1/2}/(LW)$. We compare the predictions of the solutions to the boundary-value problems with the results of physical experiments of Johnson, et al. [12] and Pouliquen [25].

3.1 Differential equations

The balances of momentum parallel and perpendicular to the flow are

$$S' = -\rho g \sin\phi + 2\mu_w p/W, \quad (17)$$

where g is the gravitational acceleration and

$$u' = S/\mu, \quad (18)$$

with μ given by (5) and (6); and

$$p' = -\rho g \cos\phi, \quad (19)$$

where, upon carrying out the differentiation and using (7), this first order equation for p can be employed as a first order equation for ν:

$$\left[T(4\rho GF)_\nu - 4\rho GF\eta/\kappa\right]\nu' = 4\rho GFQ/\kappa - \rho g\cos\phi, \quad (20)$$

in which η is given by (10) and (11) and κ is given by (8) and (9). The energy balance is

$$Q' = Su' - \Gamma - (2/\pi)^{1/2}(1-e_w^2)pT^{1/2}/(LW), \quad (21)$$

where Γ is given by (12) with d replaced L from (14) when $L/d > 1$; and

$$T' = -\left(Q + \eta\nu'\right)/\kappa, \quad (22)$$

where ν' is given in (20). The specification of the mass hold-up per unit area, M, is implemented as a boundary condition to a first order differential equation for the partial mass hold-up, $I(y) \equiv \int_0^y \rho(\xi)d\xi$:

$$I' = \rho \quad (23)$$

with $I(0) = 0$ and $I(H) = M$.

3.2 Boundary conditions

We assume that the bottom boundary consists of a flat wall to which spheres identical to those in the flow have been attached and use the boundary conditions on the slip velocity and energy flux at a bumpy, nearly elastic, frictionless boundary derived by Richman [27] with the effective coefficient of restitution.

The balance of momentum tangent to the boundary provides

$$\frac{u}{T^{1/2}} = \left(\frac{\pi}{2}\right)^{1/2} f \frac{S}{p}, \tag{24}$$

where

$$f = \frac{3}{2^{5/2}J} \frac{2^{3/2}J - 5F(1+B)\sin^2\theta}{2(1-\cos\theta)/\sin^2\theta - \cos\theta} + \frac{5F}{2^{1/2}J}, \tag{25}$$

with $B = [1 + 5/(8G)]\,\pi/(2^{1/2}12)$, and θ, the bumpiness, measures the average maximum penetration of a flow sphere between boundary spheres. When the diameter of the flow spheres is the same as that of the boundary spheres, the bumpiness is given in terms of d and the average separation s between the edges of the boundary spheres by $\sin\theta = (d+s)/(2d)$. In what follows, we employ values of θ of $\pi/5$ and $\pi/3$ that correspond to relatively smooth and relatively rough boundaries, respectively.

The balance of energy at the bumpy boundary is [27]

$$Su = Q + D, \tag{26}$$

where $D = (2/\pi)^{1/2} p T^{1/2} h(1-\varepsilon)$, with $h = 2(1-\cos\theta)/\sin^2\theta$.

We take the top of the flow to be the point where the free flight trajectory of a particle ejected normal to the flow with velocity $T^{1/2}$ first equals the mean free distance between collisions. At this point [28],

$$p = \rho T = 0.039 \rho g d/\nu. \tag{27}$$

The flow momentum and energy flux there, associated with the acceleration of the particle under gravity [29], are

$$S = p\tan\phi \tag{28}$$

and

$$Q = -pT^{1/2}\tan^2\phi. \tag{29}$$

Above the top of the flow is a rare gas of spheres in ballistic trajectories that we neglect for the sake of simplicity.

4 Algebraic Theory

The numerical simulations of Mitarai & Nakanishi [30] show that the profile of the granular temperature T in a dense inclined flow may be determined using the algebraic balance between production and dissipation of fluctuation energy. Jenkins [13] confirmed this in the context of his extension of the kinetic theory that introduced the additional length scale into the rate of collisional dissipation.

In this section, we employ the constitutive theory outlined above and the algebraic balance of the energy with the additional length scale to study flows over bumpy planes confined between parallel, vertical walls. As in the solution of the boundary-value problem, we take sliding friction of the side walls into account in an approximate way by including the average frictional resistance of the side walls in a one-dimensional analysis through the thickness of the flow . At the base, we assume that the flow slips relative to the rigid, bumpy boundary. The flow is taken to be so dense that the extension of the kinetic theory that involves the additional length scale applies throughout its depth.

From (14) and (16), L/d is greater than one throughout the flow if

$$\tan^2\phi \le \frac{24J(1-\varepsilon)}{5\pi(1+\varepsilon)} \tag{30}$$

This limits the angles of inclination for which the analysis applies. For greater angles of inclination, there is a more dilute region above the dense region that is described by the classical kinetic theory. Here we assume that the upper surface is free and show only these results.

The mass-hold-up M and the approximately constant average mass density ρ are assumed to be known, so the thickness H, of the flow, given as their ratio, is also known. In a steady, fully-developed, inclined flow in a dense layer of thickness H, the pressure p is given to a good approximation by

$$p = \rho g(H-y)\cos\phi. \tag{31}$$

Denoting quantities evaluated at the base with the subscript 0, $p_0 = \rho g H\cos\phi$. With $p = 2\rho(1+\varepsilon)GT$,

$$T = \frac{g(H-y)}{2(1+\varepsilon)G}, \tag{32}$$

and $T_0 = gH/\left[2(1+\varepsilon)G_0\right]$.

In the dense flow, (17) and (19) may be integrated to

$$\frac{S}{p} = \tan\phi - \mu_w \frac{p}{\rho g W\cos\phi}, \tag{33}$$

and, with α defined as the stress ratio S_0/p_0 at the base, $\alpha = \tan\phi - \mu_w H/W$. With this, the average value of S/p is $(\tan\phi + \alpha)/2$. In order to obtain an analytical expression for the average velocity in the dense flow, we evaluate G given by (16) at this average value and denote this value by $\overline{G}$. Then (17) and (18) yield

$$u = u_0 + \frac{2}{3}\left[1 - \left(\frac{p}{p_0}\right)^{3/2}\right] \frac{A p_0^{3/2}\tan\phi}{\rho^{3/2}\left(gd\cos\phi\right)} - \frac{2}{5}A\left[1 - \left(\frac{p}{p_0}\right)^{5/2}\right] \frac{\mu_w d}{W\cos\phi} \frac{p_0^{5/2}}{\rho^{5/2}\left(gd\cos\phi\right)^2}, \tag{34}$$

where $A^2 \equiv 25\pi(1+\varepsilon)/\left(32J^2\overline{G}\cos^2\phi\right)$, and $u_0 = f\alpha\,(\pi/2)^{1/2}T_0^{1/2}$.

The integral of u over the thickness of the flow gives the depth-averaged velocity, U, as

$$U = u_0 + \frac{2}{35}\frac{Ap_0^{3/2}\tan\phi}{\rho^{3/2}gd\cos\phi}\left(7 - 5\frac{p_0}{\rho g\cos\phi}\frac{\mu_w}{W\sin\phi}\right). \quad (35)$$

The product of the depth-averaged velocity, the flow thickness, and the constant mass density is the mass flow rate.

5 Results

We employ the two-point Matlab boundary-value problem solver 'bvp4c' to determine solutions for a range of values of the the mass hold-up at a given angle of inclination in chutes of two different widths and, in the wider chute, at two different values of the bumpiness of the base. In addition, we use the algebraic theory to predict the relationship between the mass flow rate and the mass hold-up. In what follows, all quantities are given in dimensionless form, using the particle density, ρ/ν, the particle diameter, d, and the gravitational acceleration, g.

The results are shown in Fig. 1 together with data from the experiments of Pouliquen [25]. For large enough values of mass hold-up, the algebraic solutions can reproduce both the experimental data and the results of the numerical solutions. At the smaller values of the mass hold-up, the relation between the mass flow rate and the mass hold-up is smooth, but not monotonic. As a consequence, over a range of mass flow rates, as many as three solutions can exist [20]. This is in agreement with physical experiments on inclined flows of particles over a flat, frictional base [12]. Johnson, et al. [12] observed that, for a certain range of mass flow rates, both dilute and dense flows were possible, depending on the upstream conditions.

We find that classical kinetic theory applies throughout the dilute flows and extended kinetic theory applies throughout the dense flows; an intermediate solution involves a dense core and dilute regions at its top and bottom. The three numerical solutions for such a mass flow rate are shown in Fig. 2. As expected, the algebraic solutions agree with the numerical solutions (Fig. 1) only for thick, dense flows; in these, the dense core occupies almost all of the flow. In thin flows, the boundaries create regions in which the transport of energy can not be neglected.

Our numerical solutions agree with the experimental observations of Johnson, et al. [12] in two other respects. First, the jumps that they observe between the dilute and the dense regimes can be explained by the presence of local maxima in the curves of Fig. 1. Upon increasing the mass flow rate, flows follow the dilute branch of the numerical solutions up to a point where dilute solutions are no longer possible, causing an abrupt transition to the dense branch. Second, in presence of side walls, the dense branch tends asymptotically to a limiting value of the mass flow rate.

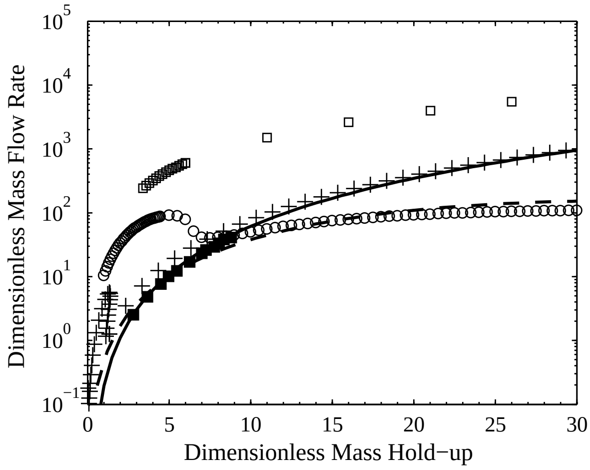

Fig. 1 Numerical solutions: open squares, $W = 1400$, $\theta = \pi/5$; plus signs, $W = 1400$, $\theta = \pi/3$; open circles, $W = 30$, $\theta = \pi/5$. Algebraic solutions: solid line, $W = 1400$, $\theta = \pi/3$; dashed line, $W = 30$, $\theta = \pi/5$. Pouliquen's experiments: filled squares. All solutions for $\phi = 26^o$, $\varepsilon = e_w = 0.60$, $\mu_w = 0.15$, and $c = 0.50$.

The hysteresis observed in the experiments of Johnson, et al. [12] provides an indication that the intermediate branch is unstable. However, the instability analysis of Woodhouse and Hogg [31] for unconfined, inclined flow indicates that the region of linear stability is located on the intermediate branch and exists only for a limited range of inclination angles. In the physical experiments, the instability of the outer branches may be suppressed by the presence of the walls, but the question of the stability of the confined flows derserves more study.

6 Conclusions

We have applied classical, extended, and algebraic kinetic theories for inelastic, frictional spheres to regimes of uniform, steady, inclined flows over a rigid, bumpy bed between flat, frictional side walls. The predicted average values of the flow variables and the predicted profiles of volume fraction and average velocity exhibit the same features as those measured in physical experiments.

References

1. Goldhirsch, I.: Rapid granular flows. Ann. Rev, Fluid Mech. 35, 267 (2003)

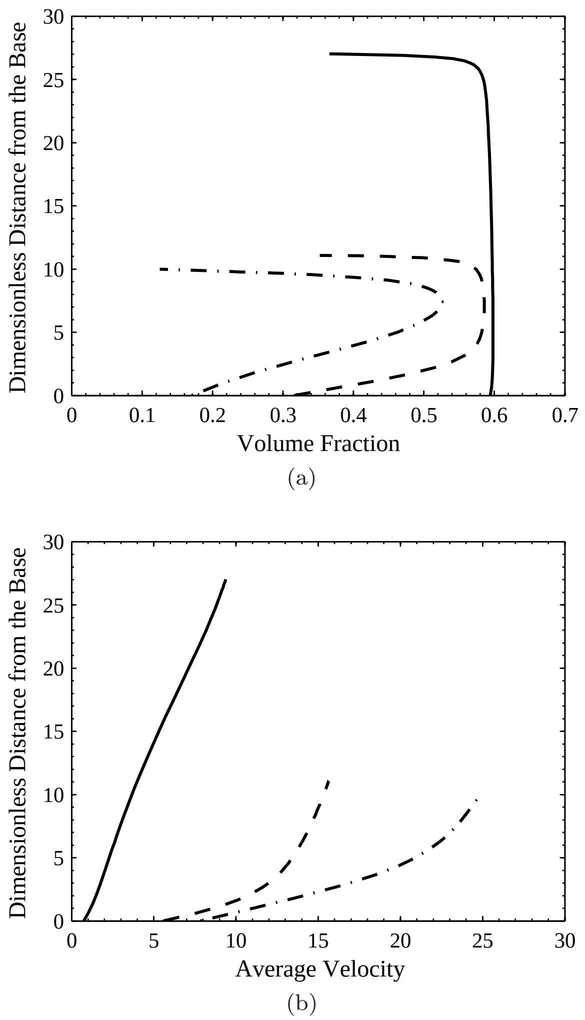

Fig. 2 Multiple numerical profiles of (a) volume fraction and (b) average velocity obtained for $\phi = 26^o$, $W = 30$, $\theta = \pi/5$, at a dimensionless mass flow rate of about 80: dot-dashed lines, dimensionless hold-up of 4; dashed lines, dimensionless hold-up of 6; solid lines, dimensionless hold-up of 16.

2. Jenkins, J.T., Richman, M.W.: Grad's 13-moment system for a dense gas of inelastic spheres. Arch. Rat. Mech. Anal. 87, 355 (1985)
3. Garzo, V., Dufty, J.W.: Dense fluid transport for inelastic hard spheres. Phys. Rev. E 59, 5895 (1999)
4. Lutsko, J.F.: Transport properties of dense dissipative hard-sphere fluids for arbitrary energy loss models. Phys. Rev. E 72, 021306 (2005)
5. Chou, C.-S., Richman, M.W.: Constitutive theory for homogeneous granular shear flows of highly inelastic spheres. Physica A 259, 430 (1998)
6. Montanero, J.M., Garzo, V., Santos, A., Brey, J.J.: Kinetic theory of simple granular shear flows of smooth hard spheres. J. Fluid Mech. 389, 391 (1999)
7. Kumaran, V.: Dense granular flow down an inclined plane: from kinetic theory to granular dynamics. J. Fluid Mech. 599, 120 (2008)
8. Lun, C.K.K.: Kinetic theory for granular flow of dense, slightly inelastic, slightly rough spheres. J. Fluid Mech. 233, 539 (1991)
9. Jenkins, J.T., Zhang, C.: Kinetic theory for identical, frictional, nearly elastic spheres. Phys. Fluids 14, 1228 (2002)
10. Kumaran, V.: Dynamics of dense, sheared granular flows. Part I. Structure and diffusion. J. Fluid Mech. 632, 109 (2009)
11. Kumaran, V.: Dynamics of dense, sheared granular flows. Part II. The relatively velocity distributions. J. Fluid Mech. 632, 145 (2009)
12. Johnson, P.C., Jackson, R.: Frictional-collisional constitutive relations for granular materials, with applications to plane shearing. J. Fluid Mech. 176, 67 (1987)
13. Jenkins, J.T.: Dense inclined flows of inelastic spheres. Gran. Matt. 10, 47 (2007)
14. Jenkins, J.T., Berzi, D.: Dense inclined flows of inelastic spheres: tests of an extension of kinetic theory. Gran. Matt. 12, 151 (2010)
15. Berzi, D., Di Prisco, C.G., Vescovi, D.: Constitutive relations for steady, dense granular flows. Phys. Rev. E 84, 031301 (2011).
16. Azanza, E., Chevoir, F., Moucheront, P.: Experimental study of collisional granular flows down an inclined plane. J. Fluid Mech. 400, 199 (1999)
17. Forterre, Y., Pouliquen, O.: Longitudinal vortices in granular fows. Phys. Rev. Lett. 86, 5886 (2001)
18. Anderson, K.G., Jackson, R.: A comparison of the solutions of some proposed equations of motion of granular materials for fully developed flow down inclined planes. J. Fluid Mech. 241, 145 (1992)
19. Berzi, D., Jenkins, J.T.: Surface flows of inelastic spheres. Phys. Fluids 23, 013303 (2011)
20. Woodhouse, M.J., Hogg, A.J., Stellar, A.A.: Rapid granular flow down inclined planar chutes. Part 1. Steady flows, multiple solutions, and existence domains. J. Fluid. Mech. 652, 427 (2010)
21. Lun, C.K.K., Bent, A.A.: Numerical simulation of inelastic frictional spheres in simple shear flow. J. Fluid Mech. 258, 335 (1994)
22. Herbst, O., Huthmann, M., Zippelius, A.: Dynamics of inelastically colliding spheres with Coulomb friction: Dynamics of the relaxation of translational and rotational energy. Gran. Matt. 2, 211 (2000)
23. Carnahan, N.F., Starling, K.E.: Equations of state of non-attracting rigid spheres. J. Chem. Phys. 51, 635 (1969)
24. Foerster, S.F., Louge, M.Y., Chang, H., Allia, K.: Measurements of Collision properties of small spheres. Phys. Fluids 6, 1108 (1994)
25. Pouliquen, O.: Scaling laws in granular flows down a bumpy inclined plane. Phys. Fluids 11, 542 (1999)
26. Torquato, S.: Nearest-neighbor statistics for packings of hard spheres and disks. Phys. Rev. E 51, 3170 (1995)
27. Richman, M.W.: Boundary conditions based on a modified Maxwellian velocity distribution function for flows of identical, smooth, nearly elastic spheres. Acta Mech. 75, 227 (1988)
28. Pasini, J.M., Jenkins, J.T.: Aeolian transport with collisional suspension. Phil. Trans. Roy. Soc. 363, 1625 (2005)
29. Jenkins, J.T., Hanes, D.M.: The balance of momentum and energy at an interface between colliding and freely flying grains in a rapid granular flow. Phys. Fluids A 5, 781 (1993)
30. Mitarai, N., Nakanishi, H.: Bagnold scaling, density plateau, and kinetic theory analysis of dense granular flow. Phys. Rev. Lett. 94, 128001 (2005)
31. Woodhouse, M.J. and Hogg, A.J.: Rapid granular flow down inclined planar chutes. Part 2. Linear stability analysis of steady flow solutions. J. Fluid. Mech. 652, 461 (2010)